M.Math. IInd year : Commutative Algebra
Midsemestral Exam : Instructor : B. Sury
September 22, 2022
Answer Q 1 and FIVE MORE questions.
Q 1 carries 10 points and all other questions carry 5 points each.
Be BRIEF (e.g.: for each part of Q 1, give a solution in 1-2 lines).
Maximum Marks 35.
In what follows, A is a commutative ring with $1 \neq 0$.

Q 1. (10 points)
(a) *Simplify* $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_n$.
(b) *Prove that* $\mathbb{Q}^{>0}$ *is a free abelian group.*
(c) *Show that* $A[X]$ *is* A*-flat.*
(d) *Prove that* $A[X]$ *is not a local ring.*
(e) *Let* $a^n = a$ *for all* $a \in A$. *Prove every prime ideal is maximal.*

Q 2. *Let* I *be an ideal of* A *which is a direct summand. Show that* I *is a principal ideal generated by an idempotent element. Further, in this case, show that every short exact sequence of the form*

$$0 \to N \to M \to I \to 0$$

splits.

Q 3. *Let* M *be a flat* A*-module, and suppose*

$$0 \to M_1 \to M_0 \to M \to 0$$

is an exact sequence. Show that for any A*-module* N, *the sequence*

$$0 \to M_1 \otimes N \to M_0 \otimes N \to M \otimes N \to 0$$

is exact.
Hint. Consider a surjection $f : F \to N$ from F free and tensor the given exact sequence with $Ker(f), F, N$ and apply snake's lemma.

Q 4. *If* M *is finitely generated and* $\theta : M \to M$ *is a surjection, prove that* θ *must be injective. Using this, prove that if* $A^m \cong A^n$, *then* $m = n$.

Q 5. Consider the $\mathbb{Z}$-module $M = \oplus_{n\geq 2}\mathbb{Z}_n$ and consider $S = \mathbb{Z} - \{0\}$. Determine $S^{-1}ann(M)$ and $annS^{-1}M$.

Q 6. *Let M, N be A-modules where N is finitely generated, and let J be the Jacobson radical of A. If $f : M \to N$ is a A-homomorphism such that the induced map $\overline{f} : M/JM \to N/JN$ is surjective. Prove that f is surjective.*
Hint. Think of NAK lemma for a suitable module.

Q 7. *If $0 \to M \to A^n \to N \to 0$ is an exact sequence, and Q is any A-module, prove that $Tor_i(M, Q) \cong Tor_i(N, Q)$ for all $i \geq 1$.*
Hint. Use the long exact sequence for Tor.

Q 8. *If A/J is isomorphic to a product of finitely many fields (where J is the Jacobson radical of A), then prove that A has only finitely maximal ideals.*
Hint. Think of the Chinese remainder theorem.